NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

Year 12

Trial Examination

2021

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Write your name on the front of every booklet.
- In Questions 11 to 14 show relevant mathematical reasoning and/or calculations.
- NESA approved calculators may be used.
- Weighting:30%

Section I Multiple Choice

- 10 marks
- Attempt all questions.
- Answer Sheet provided
- Allow about 15 minutes for this section

Section II Free Response

- 60 marks
- Start a separate booklet for each question.
- Each question is of equal value.
- All necessary working should be shown in every question.
- Allow about 1 hour and 45 minutes for this section.

Section I

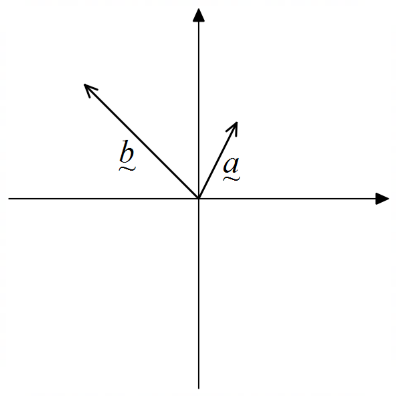
10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

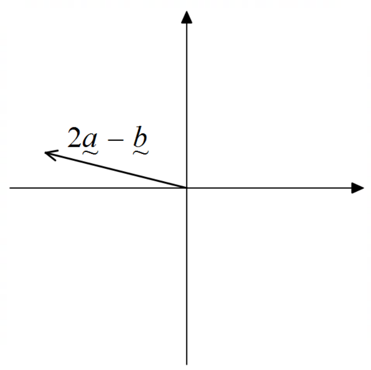
Use the multiple choice answer sheet for Questions 1 – 10.

Q1. Vectors $\underset{\sim}{\mathbf{a}}$ and $\underset{\sim}{\mathbf{b}}$ and are shown in the diagram below.

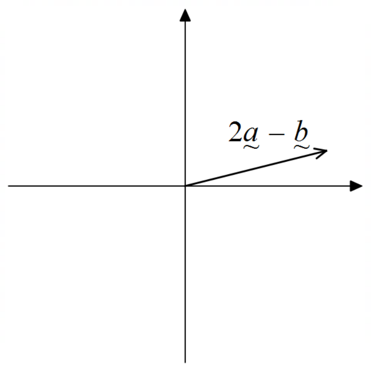


Which diagram best represents the vector $2\underset{\sim}{\mathbf{a}} - \underset{\sim}{\mathbf{b}}$?

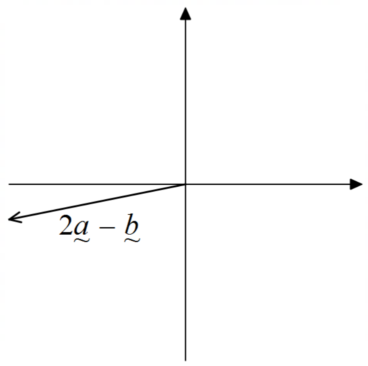
A



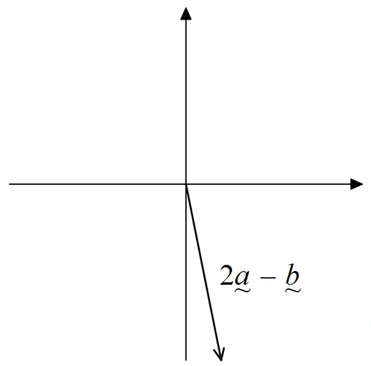
B



C



D



Q2. The radius of a circle is increasing at a constant rate of 0.4 meters per second. What is the rate of increase in the area of the circle at the instant when the circumference is 60π metres?

A. $0.24 \frac{m^2}{\text{sec}}$

B. $24 \frac{m^2}{\text{sec}}$

C. $24\pi \frac{m^2}{\text{sec}}$

D. $240\pi \frac{m^2}{\text{sec}}$

Q3. Which of the following expressions is equivalent to $3\cos x - \sqrt{3}\sin x$?

A. $2\sqrt{3}\cos\left(x - \frac{\pi}{3}\right)$

B. $2\sqrt{3}\cos\left(x + \frac{\pi}{3}\right)$

C. $2\sqrt{3}\cos\left(x - \frac{\pi}{6}\right)$

D. $2\sqrt{3}\cos\left(x + \frac{\pi}{6}\right)$

Q4. Find the equation to the tangent to the curve given by $x = 2y$ and $y = t^2 + 5$ at the point where $t = 2$

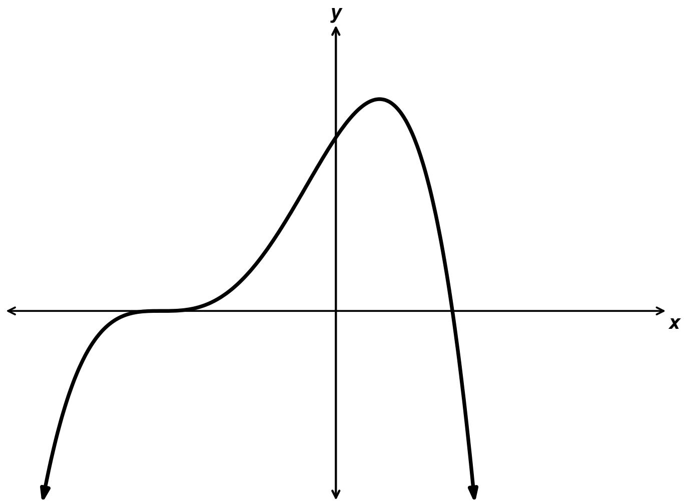
A. $y = 2x + 1$

B. $y = tx - 2t + 6$

C. $y = x + 4$

D. $y = x - 4$

Q5. The graph of a polynomial $y = P(x)$ is shown below.



What is the minimum possible degree of the polynomial $[P(x)]^2$?

A. 2

B. 4

C. 8

D. 16

Q6. The variance of a Bernoulli distribution is $\dfrac{3}{16}$.

What is a possible value for the mean of the distribution?

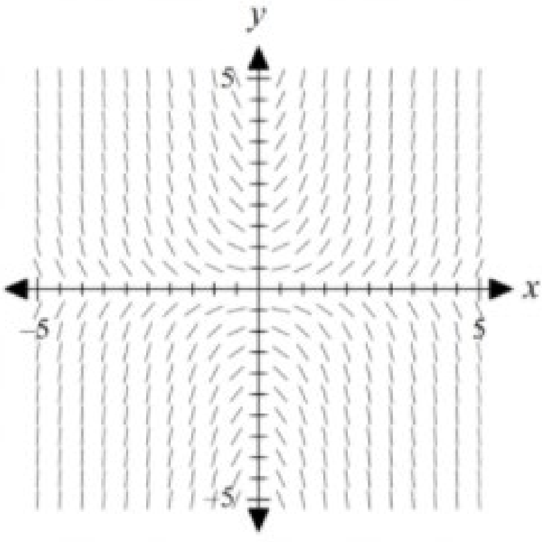
A. $\dfrac{5}{4}$

B. $\dfrac{3}{4}$

C. $\dfrac{3}{16}$

D. $\dfrac{5}{16}$

Q7. The direction field of a differential equation is shown.



Which differential equation does the direction field best represent?

A. $y' = \dfrac{x}{y}$

B. $y' = xy$

C. $y' = x^2 y^2$

D. $y' = x + y$

Q8. Which of the following shows the correct substitution of the variable u into the integral

$\int x\sqrt{3x - 2}\, dx$ using $u = 3x - 2$?

A. $\int x\left(u^{\frac{1}{2}}\right) dx$

B. $\int \left(\dfrac{u + 2}{3}\right) u^{\frac{1}{2}}\, \mathrm{du}$

C. $\dfrac{1}{9}\int (u + 2)\, u^{\frac{1}{2}}\, \mathrm{du}$

D. $\dfrac{1}{3}\int \left(\dfrac{u}{3} + 2\right) u^{\frac{1}{2}}\, \mathrm{du}$

Q9. The region bounded by the y-axis and the curve $y = 6x - x^2 : D[0,3]$ is rotated around the y-axis. The volume of the resulting solid of revolution is given by:

A. $\displaystyle\int_0^3 6x - x^2 \, dx$

B. $\displaystyle\int_0^3 \pi(6x - x^2)^2 \, dx$

C. $\displaystyle\int_0^9 \pi(3 + \sqrt{9 - y})^2 \, dy$

D. $\displaystyle\int_0^9 \pi(3 - \sqrt{9 - y})^2 \, dy$

Q10. The scalar projection of the vector $\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{i}} + 2\underset{\sim}{\mathbf{j}}$ in the direction of the vector $\underset{\sim}{\mathbf{v}}$ is 2.

Which of the following could be equal to $\underset{\sim}{\mathbf{v}}$?

A. $3\underset{\sim}{\mathbf{i}} + 4\underset{\sim}{\mathbf{j}}$

B. $3\underset{\sim}{\mathbf{i}} + 5\underset{\sim}{\mathbf{j}}$

C. $4\underset{\sim}{\mathbf{i}} + 3\underset{\sim}{\mathbf{j}}$

D. $4\underset{\sim}{\mathbf{i}} + 5\underset{\sim}{\mathbf{j}}$

End of Multiple Choice

Section II

60 marks - Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

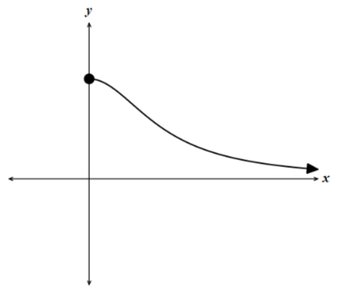
In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11: Use new writing book for this question **15 marks**

a) Solve for x given $\frac{x}{x-3} \geq 4$ **2**

b) Given $\cos x = -\frac{1}{3}$ and $\sin x > 0$ find the exact value of $\sin 2x$ **2**

c) The diagram shows the function $f(x) = \frac{1}{x^2+1}$ for $x \geq 0$.



State the domain and range of the inverse function $f^{-1}(x)$. **2**

d) Solve for x given $x^3 - 5x^2 + 8x - 4 = 0$ **3**

e) The ten members of a sporting team have the numbers 1 to 10 on the back of their jumpers.

i. Explain why, if six players are randomly chosen there must be at least one pair of players whose numbers add to 11. **1**

ii. What is the minimum number of players that need to be chosen to ensure that at least one pair of players have numbers that add to 12? **2**

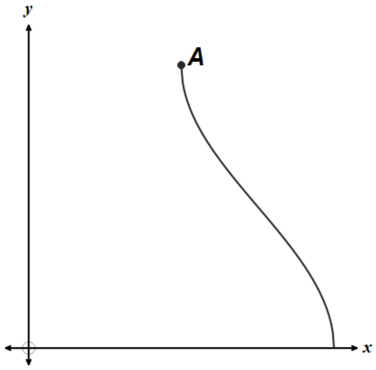
f) Use mathematical induction to prove **3**

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \quad \text{for integers } n \geq 1$$

End of Question 11

Question 12 : Use new writing book for this question **15 marks**

a) The graph above below $y = \cos^{-1}(2x - 3)$. Determine the coordinates of point A. **2**



b) In a dice game five dice are rolled simultaneously. One point is scored for each six showing face-up on the dice. Let X represent the score.

(i) Explain why X can be considered as a Bernoulli variable **1**

(ii) Find the expected value and variance of X **1**

(iii) Find $P\,(X > 3)$, correct to four decimal places. **1**

c) Find $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 2x \; dx$ in simplest exact form. **3**

d) Using the substitution $x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, determine the exact value of

$$\int_0^1 \frac{1}{\sqrt{4 - x^2}}\, dx$$ **3**

e) Find the solution of the differential equation $y' = 1 + 4y^2$ given $y(0) = \frac{1}{2}$ 4

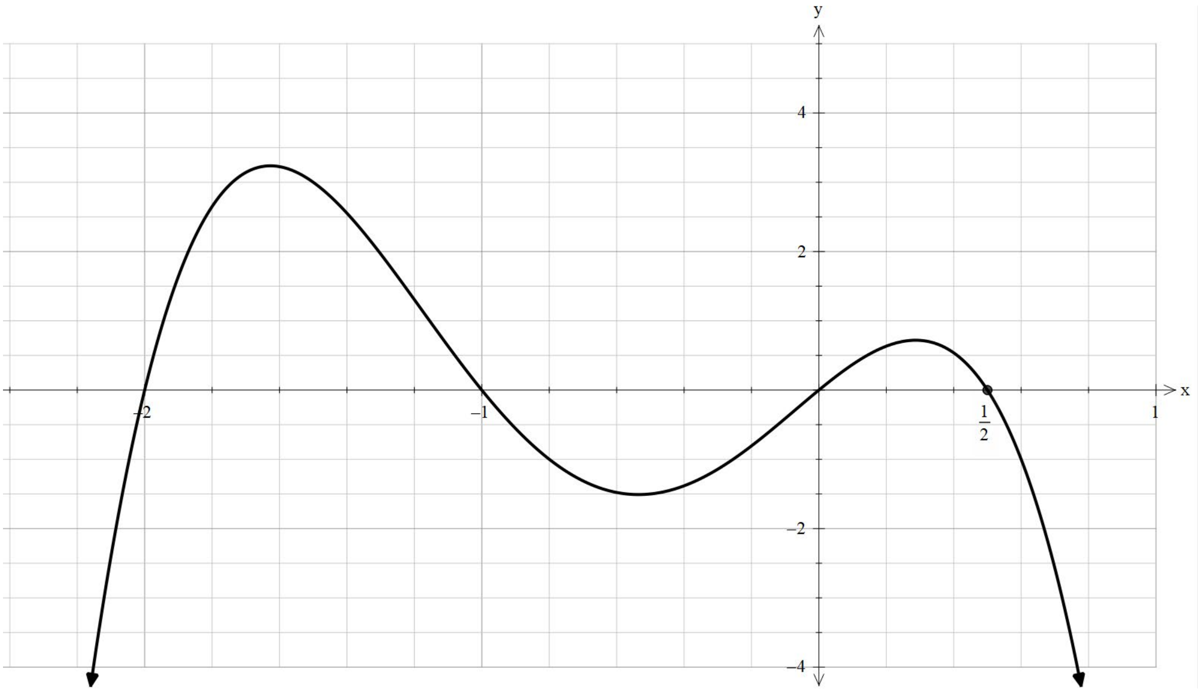
End of Question 12

Question 13 : Use new writing book for this question **15 marks**

a) The curve $y = f(x)$ is shown below.
Using the Cartesian planes provided in the answer booklet, sketch the following graphs.

i. $y = f(|x|)$ **1**

ii. $y^2 = f(x)$ **3**



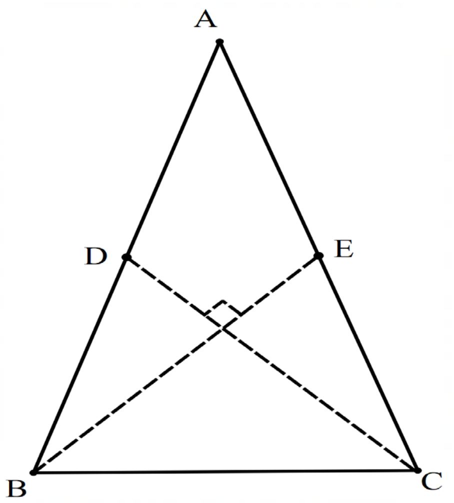
b) Shares in a company X are particularly volatile. On any given trading day the probability that the share price will increase by \$1 is $\frac{3}{5}$ and the probability that the share price will fall by \$1 is $\frac{2}{5}$. If the share price is now \$6 find the probability that it will be less than \$6 after 5 trading days. **2**

Question 13 continues on next page;

Question 13 continued.

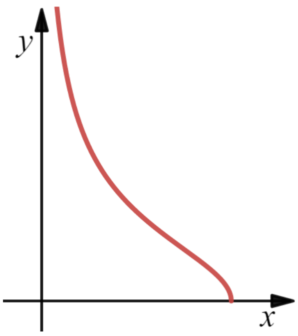
c) Determine the equation to the tangent to the curve $y = 3\sin^{-1}\left(\frac{x}{2}\right)$ at $x = 1$ **3**

d) In the isosceles triangle ABC, $|\overrightarrow{AB}| = |\overrightarrow{AC}|$. D is the midpoint of side AB and E is the midpoint of side AC. $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{BE}$.



Use **<u>vector methods</u>** to find the size of $\angle BAC$. **3**

e) Part of the graph of the function $f(x) = \sqrt{\dfrac{1 - 3x^2}{4x^2}}$ is shown below.



The region bounded by the curve $y = f(x)$, $y = \frac{3}{2}$ and the coordinate axes is rotated about the y-axis. Find the exact volume of the solid formed. **3**

End of Question 13.

Question 14 : Use new writing book for this question **15 Marks**

a) i. Show $\sin(x - y)\sin(x + y) = \sin^2 x - \sin^2 y$. **2**

ii. Solve $\sin^2 3x - \sin^2 x = \sin 4x$ for $0 \leq x \leq \pi$ **2**

b) The population $P(t)$ of bacteria in a petri dish is modelled by the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{6}\left(1 - \frac{P}{8000}\right)$$ where $P(0)=P_0$ and t is the time in hours.

i. If the initial population P_0 is 1000 bacteria, show that $P(t) = \dfrac{8000}{1 + 7e^{-\frac{t}{6}}}$ **3**

(You may use the fact that $\dfrac{Q}{P(Q - P)} = \dfrac{1}{P} + \dfrac{1}{Q - P}$)

ii. If instead the initial population P_0 were 12000 bacteria, describe what would have happened to the population as t increases. **1**

c) The vectors $\underset{\sim}{\mathbf{p}} = \begin{pmatrix} a \\ 3 \end{pmatrix}$ and $\underset{\sim}{\mathbf{q}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are parallel.

i. Find a. **1**

ii. The vector $\underset{\sim}{\mathbf{r}} = \begin{pmatrix} x \\ y \end{pmatrix}$ is perpendicular to vector $\underset{\sim}{\mathbf{q}}$ and $\left|\underset{\sim}{\mathbf{q}} - \underset{\sim}{\mathbf{r}}\right| = \sqrt{65}$.

Find the possible values of x and y. **3**

d) Using the expansion of $(1 + x)^{2n}$ and given that **3**

$${}^{2n}\mathbf{C}_1 + 2({}^{2n}\mathbf{C}_2) + 3({}^{2n}\mathbf{C}_3) + \ldots + 2n({}^{2n}\mathbf{C}_{2n}) = n \times 2^{2n}$$

Show that

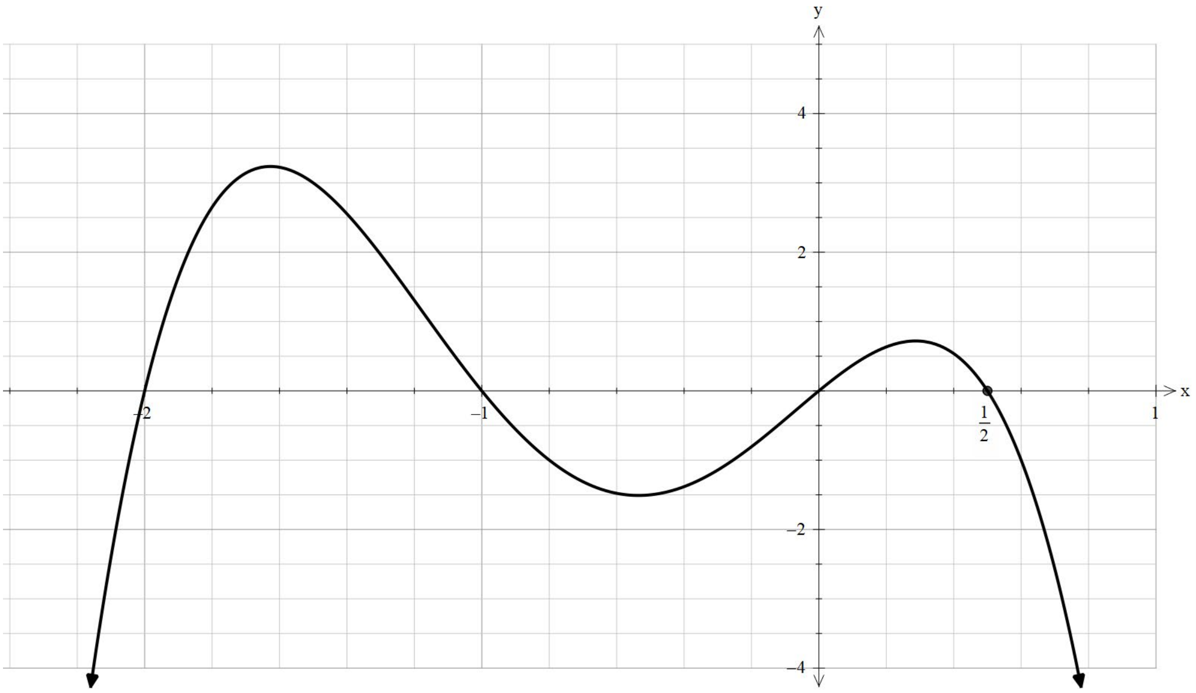
$$2({}^{2n}\mathbf{C}_1) + 3({}^{2n}\mathbf{C}_2) + 4({}^{2n}\mathbf{C}_3) + \ldots + (2n + 1)({}^{2n}\mathbf{C}_{2n}) = (n + 1) \times 2^{2n} - 1$$

End of Examination

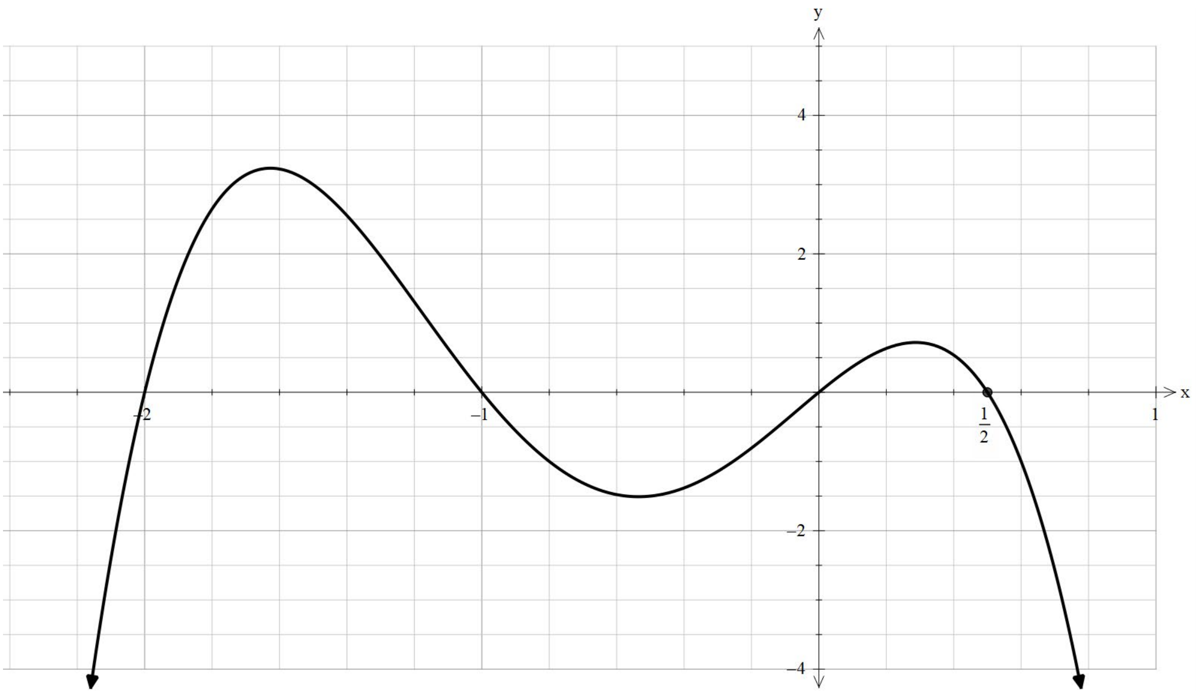
Student Number:

Answer sheet for Question 13b.

i. $y = f(|x|)$ 1



ii. $y^2 = f(x)$ **3**



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Question 11: Use new writing book for this question **15 marks**

a) Solve for x given $\frac{x}{x-3} \geq 4$ **2**

$$\frac{x}{x-3} \geq 4$$
$$x \neq 3$$

$$\frac{x}{x-3} = 4$$
$$x = 4x - 12$$
$$3x = 12$$
$$x = 4$$

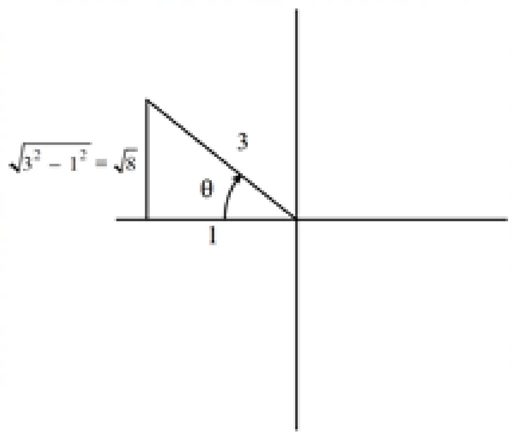
Test regions

$3 < x \leq 4$ 2 marks correct answer , one mark correct significant points

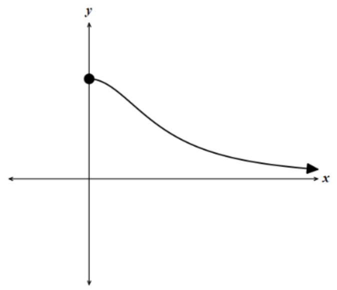
b) Given $\cos x = -\frac{1}{3}$ and $\sin x > 0$ find the exact value of $\sin 2x$ **2**

Solution

$$\sin 2x = 2\cos x \sin x$$
$$= 2 \times -\frac{1}{3} \times \frac{\sqrt{8}}{3}$$
$$= -\frac{2\sqrt{8}}{9}$$
$$= -\frac{4\sqrt{2}}{9}$$

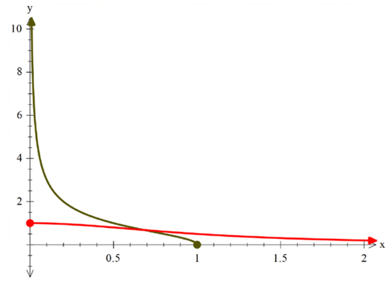


c) The diagram shows the function $f(x) = \dfrac{1}{x^2 + 1}$ for $x \geq 0$.



State the domain and range of the inverse function $f^{-1}(x)$. **2**

$$D : (0,1] \quad R : [0, \infty)$$



d) Solve for x given $x^3 - 5x^2 + 8x - 4 = 0$ **3**

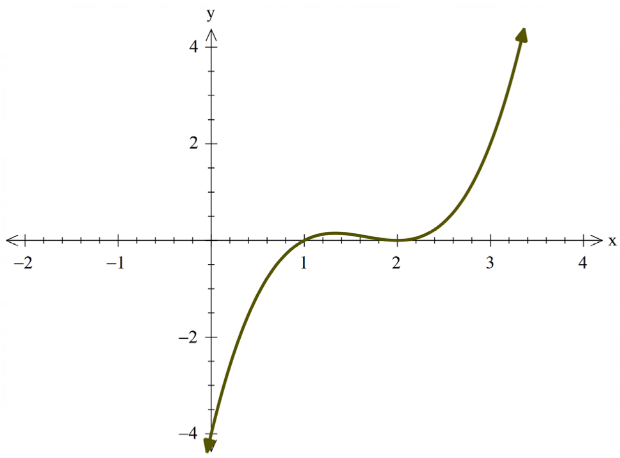
$P(1) = 1 - 5 + 8 - 4 = 0 \Rightarrow (x - 1)$ factor
$P(2) = 8 - 20 + 16 - 4 = 0 \Rightarrow (x - 2)$ factor

$P(x) = (x - 1)(x - 2)(x \pm \mathrm{c})$

$\therefore \; -4 = -1 \times -2 \times c$
$c = -2$

$P(x) = (x - 1)(x - 2)^2$

$x = 1$ or 2



e) The ten members of a sporting team have the numbers 1 to 10 on the back of their jumpers.

i. Explain why, if six players are randomly chosen there must be at least one pair of players whose numbers add to 11. **1**

ii. What is the minimum number of players that need to be chosen to ensure that at least one pair of players have numbers that add to 12? **2**

Pairs that make 11

(1,10) (2,9) (3,8) (4,7) (5,6) - if the first 5 numbers are picked one from each pair to avoid the sum of 11, the 6^{th} number must complete a pair adding to 11.

(2,10) (3,9) (4,8) (1,11) (5,7) with 1 and 6 remaining

To avoid a pair – max number would be 1, 6 plus one each from the pairs = 2+5=7

Therefore one more selection must produce a sum of 12. = 8

f) Use mathematical induction to prove **3**

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \quad \text{for integers } n \geq 1$$

Sample answer:

Step 1 Prove true for $n = 1$

$LHS = \frac{1}{2}$

$RHS = 2 - \frac{1+2}{2} = \frac{1}{2}$

$\therefore$ true for $n = 1$

Step 2 Let $n = k$ be a value for which the statement is true ie

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots\dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

Step 3 Prove true for $n = k+1$

i.e prove

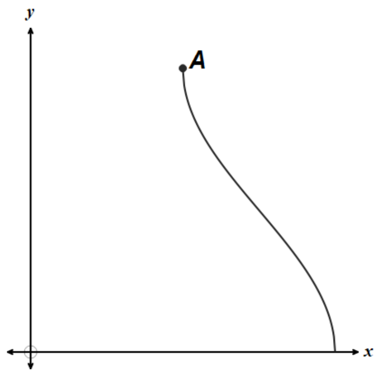
$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots\dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{(k+1)+2}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$$

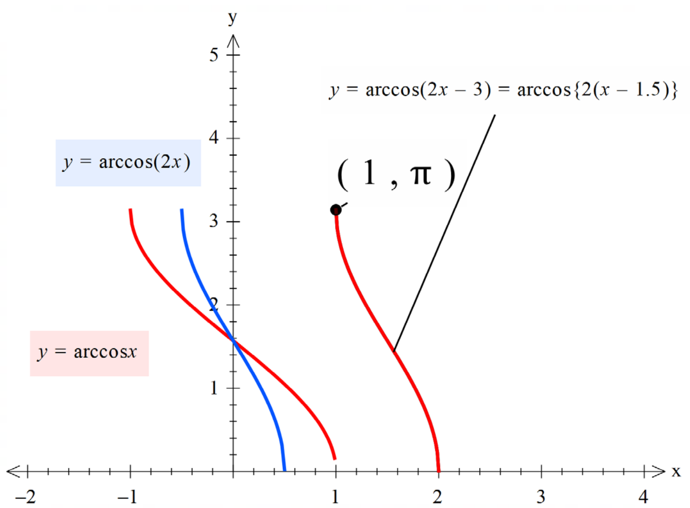
$$\begin{aligned} LHS &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \\ &= 2 - \left(\frac{2(k+2)}{2^{k+1}} - \frac{k+1}{2^{k+1}} \right) \\ &= 2 - \left(\frac{2k+4-k-1}{2^{k+1}} \right) \\ &= 2 - \frac{k+3}{2^{k+1}} \text{ as required.} \end{aligned}$$

$\therefore$ true by mathematical induction

Question 12 : Use new writing book for this question **15 marks**

a) The graph above below $y = \cos^{-1}(2x - 3)$. Determine the coordinates of point A. **2**





b) In a dice game five dice are rolled simultaneously. One point is scored for each six showing face-up on the dice. Let X represent the score.

(i) Explain why X can be considered as a Bernoulli variable **1**

(ii) Find the expected value and variance of X **1**

(iii) Find $P(X > 3)$, correct to four decimal places. **1**

X has a binomial distribution. Its shape is positively skewed.
$E(X) = np = 5 \times \frac{1}{6} = \frac{5}{6}$ (see Reference Sheet) $\text{Var}(X) = np(1-p) = 5 \times \frac{1}{6} \times \frac{5}{6} = \frac{25}{36}$ (see Reference Sheet)
$P(X > 3) = P(X = 4 \text{ or } X = 5) = P(X = 4) + P(X = 5)$ $= \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1 + \binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0$ or $\binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1 + \left(\frac{1}{6}\right)^5$ (see Reference Sheet) $= 0.0033$ (4 d.p.)

c) Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 2x \, dx$ in simplest exact form. **3**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 2x \, dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 - \cos^4 x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{1}{4}\sin\frac{4\pi}{3}\right) - \left(\frac{\pi}{4} - \frac{1}{4}\sin\pi\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{4} \times \frac{\sqrt{3}}{2}\right)$$

d) Using the substitution $x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, determine the exact value of

$$\int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx$$ **3**

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{4 - x^2}}\,dx$$

$$x = 2\sin\theta$$
$$x^2 = 4\sin^2\theta$$

$$x = 0 \Rightarrow 0 = 2\sin\theta \Rightarrow \theta = 0$$
$$x = 1 \Rightarrow \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}$$

$$dx = 2\cos\theta\,d\theta$$

$$\int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4 - 4\sin^2\theta}} \times 2\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\frac{1}{\sqrt{1 - \sin^2\theta}}2\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2} \times \frac{1}{\cos\theta} \times 2\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{6}} 1\ d\theta$$

$$= [\theta]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6}$$

e) Find the solution of the differential equation given $y(0) = \frac{1}{2}$ 4

$$y' = 1 + 4y^2$$

$$\frac{dy}{dx} = 1 + 4y^2$$

$$\int \frac{1}{1 + 4y^2}\, dy = in\ dx$$

$$\tan^{-1}(2y) = x + c$$

$$x = 0 \Rightarrow y = \frac{1}{2}$$

$$\tan^{-1}\left(2 \times \frac{1}{2}\right) = 0 + c$$

$$c = \frac{\pi}{4}$$

$$2y = \tan\left(x + \frac{\pi}{4}\right)$$

$$y = \frac{1}{2}\tan\left(x + \frac{\pi}{4}\right)$$

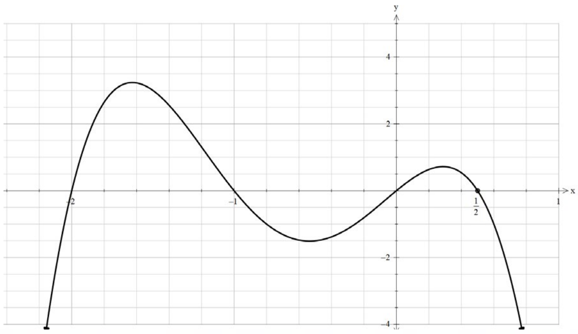
End of Question 12

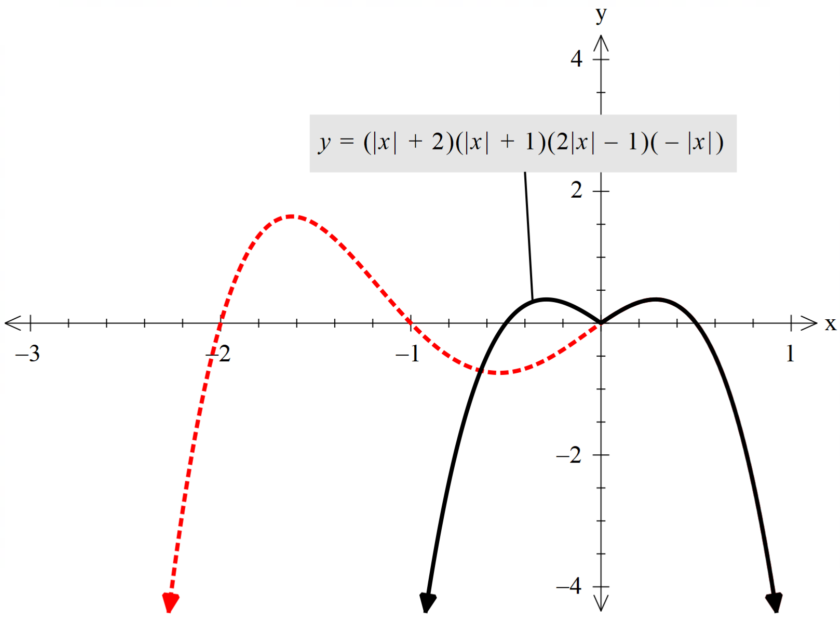
Question 13 : Use new writing book for this question **15 marks**

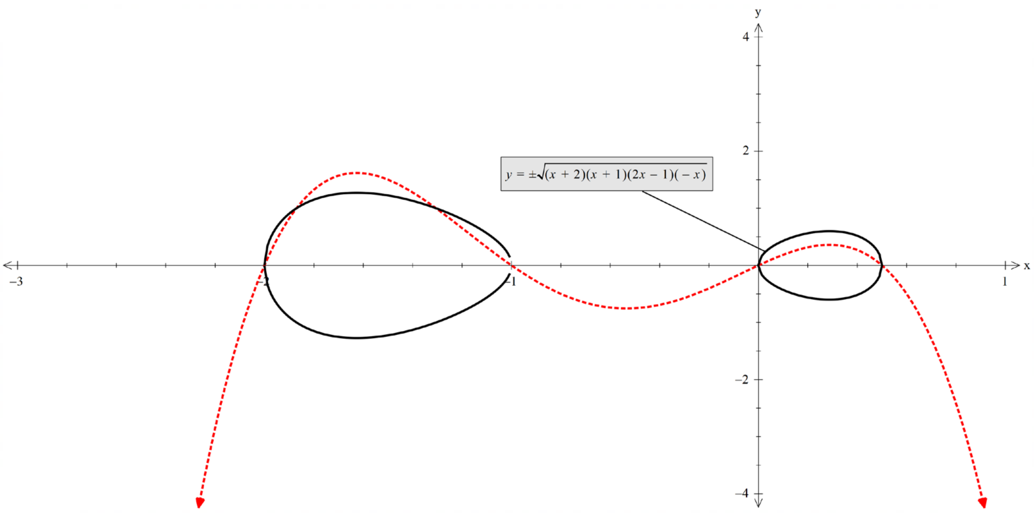
a) The curve $y = f(x)$ is shown below.
Using the Cartesian planes provided in the answer booklet, sketch the following graphs.

i. $y = f(|x|)$ **1**

ii. $y^2 = f(x)$ **3**







b) Shares in a company X are particularly volatile. On any given trading day the probability that the share price will increase by \$1 is $\frac{3}{5}$ and the probability that the share price will fall by \$1 is $\frac{2}{5}$. If the share price is now \$6 find the probability that it will be less than \$6 after 5 trading days. **2**

Criteria	Marks
• Provides correct answer.	2
• Some attempt to write in binomial probability form or equivalent progress.	1

Sample answer:

$$\begin{aligned}\text{Prob}(<\$6 \text{ after 5 days}) &= \text{Prob}(\text{Majority of movements are down})\\ &= \text{Prob}(\text{number of up movements } \le 2)\\ &= \left(\frac{2}{5}\right)^5 + {}^5C_1\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^1 + {}^5C_2\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^2\\ &= \frac{992}{3125}\end{aligned}$$

Let p(Increase) = $\frac{3}{5}$

Let q(Decrease)= $\frac{2}{5}$

Let n = no of trials

$(p+q)^n$

To be less than 6 dollars after 5 trading days must decrease on 3 or more days.

ie

Case 1 $\Rightarrow$ IIIII = \$5 increase
Case 2 $\Rightarrow$ IIIID = \$3 increase
Case 3 $\Rightarrow$ IIIDD = \$1 increase
Case 4 $\Rightarrow$ IIDDD = \$1 Dccrease
Case5 $\Rightarrow$ *IDDDD* = \$3 decrease
Case 6 $\Rightarrow$ *DDDDD* = \$5 Decrease

$$(p+q)^5 = {}^5\mathbf{C}_0\left(\frac{3}{5}\right)^5 + {}^5\mathbf{C}_1\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) + {}^5\mathbf{C}_2\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 + {}^5\mathbf{C}_3\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 + {}^5\mathbf{C}_4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^4 + {}^5\mathbf{C}_5\left(\frac{2}{5}\right)^5$$

$$P(<\$6) = {}^5\mathbf{C}_3\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 + {}^5\mathbf{C}_4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^4 + {}^5\mathbf{C}_5\left(\frac{2}{5}\right)^5$$

Question 13 continued.

c) Determine the equation to the tangent to the curve $y = 3\sin^{-1}\left(\frac{x}{2}\right)$ at $x = 1$ **3**

Determine the equation to the tangent to the curve $y = 3\sin^{-1}\left(\frac{x}{2}\right)$ at $x = 1$

$$y = 3\sin^{-1}\left(\frac{x}{2}\right) \text{ at } x = 1$$

at $x = 1$

$$y = 3\sin^{-1}\frac{1}{2} = 3 \times \frac{\pi}{6} = \frac{\pi}{2}$$

$$\frac{d\left(\sin^{-1}\frac{f(x)}{a}\right)}{dx} = \frac{f'x}{\sqrt{a^2 - [f(x)]^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{4 - x^2}}$$

at $x = 1$ $m = \frac{3}{\sqrt{3}} = \sqrt{3}$

$$y - y_1 = \sqrt{3}\,(x - x_1)$$

$$y - \frac{\pi}{2} = \sqrt{3}\,(x - 1)$$

$$y = \sqrt{3}\,x + \left(\frac{\pi}{2} - \sqrt{3}\right)$$

3 marks – correct solution

2 marks

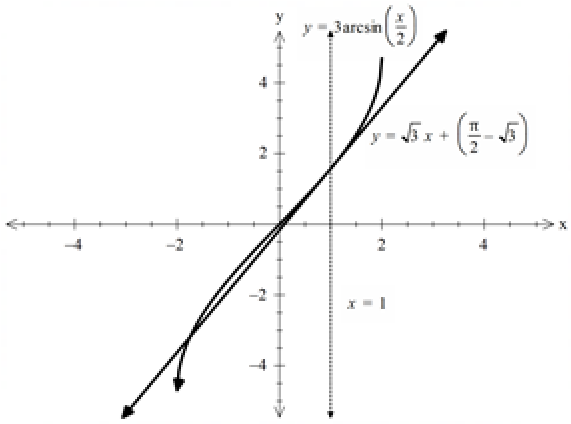
- Correct gradient from correct derivative but no further progress.
- Correct equation from a gradient correctly formed from incorrect derivative.

1 mark

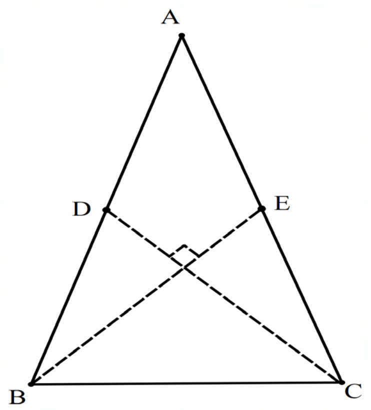
Any of

Correct y

Correct derivative



d) In the isosceles triangle ABC, $|\overrightarrow{AB}| = |\overrightarrow{AC}|$. D is the midpoint of side AB and E is the midpoint of side AC. $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{BE}$.



Use **<u>vector methods</u>** to find the size of $\angle BAC$. **3**

Let

$$\overrightarrow{AB} = \underset{\sim}{a}$$
$$\overrightarrow{AC} = \underset{\sim}{b}$$

$$|\underset{\sim}{a}| = |\underset{\sim}{b}| = x$$

$$\overrightarrow{ED} = \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$$
$$\overrightarrow{DC} = \underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$$

$$\overrightarrow{ED}.\overrightarrow{DC} = 0 \text{ as vectors perpendicular}$$

$$\left(\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}\right).\left(\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}\right) = \underset{\sim}{a} . \underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}.\underset{\sim}{a} - \underset{\sim}{b}.\frac{1}{2}\underset{\sim}{b} + \frac{1}{4}\underset{\sim}{a}.\underset{\sim}{b}$$

$$\underset{\sim}{a} . \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta = |\underset{\sim}{a}|^2 \cos\theta = x^2 \cos\theta$$

$$\underset{\sim}{a} . \underset{\sim}{a} = |\underset{\sim}{a}|^2 = x^2$$

$$\underset{\sim}{b} . \underset{\sim}{b} = |\underset{\sim}{b}|^2 = x^2$$

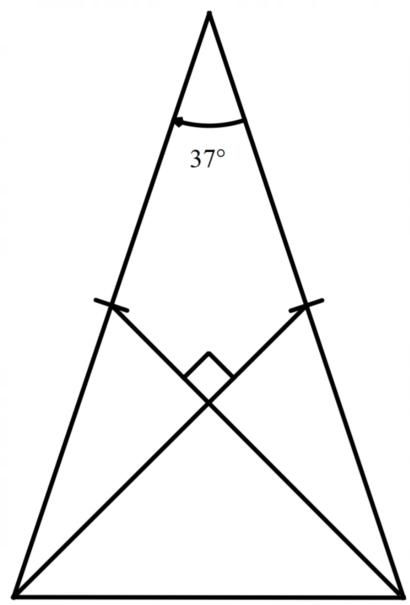
$$\frac{5}{4} \underset{\sim}{a} . \underset{\sim}{b} - \frac{1}{2} \underset{\sim}{a} . \underset{\sim}{a} - \frac{1}{2} \underset{\sim}{b} . \underset{\sim}{b} = 0$$

$$\frac{5}{4} x^2 \cos\theta - x^2 = 0$$

$$\frac{5}{4} \cos\theta - 1 = 0$$

$$\cos\theta = \frac{4}{5}$$

$$\theta = \cos^{-1} \frac{4}{5} = 36.87°$$



3 marks – full solution

2 marks – using dot product to achieve this equation or similar

$$\frac{5}{4} \underset{\sim}{a} . \underset{\sim}{b} - \frac{1}{2} \underset{\sim}{a} . \underset{\sim}{a} - \frac{1}{2} \underset{\sim}{b} . \underset{\sim}{b} = 0$$

1 mark

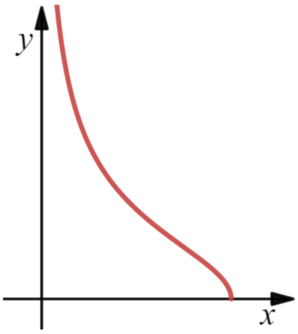
one of

- Recognition ED.DC = 0 as perpendicular
- Show $\overrightarrow{ED} = \underset{\sim}{a} - \frac{1}{2} \underset{\sim}{b}$
- Show $\underset{\sim}{a} . \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta = |\underset{\sim}{a}|^2 \cos\theta$

0 marks

Non-vector solution

iPart of the graph of the function $f(x) = \sqrt{\dfrac{1-3x^2}{4x^2}}$ is shown below.



The region bounded by the curve $y = f(x)$, $y = \dfrac{3}{2}$ and the coordinate axes is rotated about the y-axis. Find the exact volume of the solid formed.

Criteria	Marks
• Provides correct solution	3
• Finds the inverse-trigonometric primitive of the integral, or equivalent merit	2
• Determines integral that yields volume, or equivalent merit	1

Sample answer:

$y = \sqrt{\dfrac{1-3x^2}{4x^2}}$

$y^2 = \dfrac{1-3x^2}{4x^2}$

$4x^2y^2 = 1 - 3x^2$

$x^2(4y^2+3) = 1$

$x^2 = \dfrac{1}{4y^2+3}$

Now, $V = \pi \int_0^{\frac{3}{2}} x^2 dy$

$\therefore V = \pi \displaystyle\int_0^{\frac{3}{2}} \frac{1}{4y^2+3} dy$

$= \dfrac{\pi}{2\sqrt{3}} \left[\tan^{-1}\dfrac{2y}{\sqrt{3}}\right]_0^{\frac{3}{2}}$

$= \dfrac{\pi}{2\sqrt{3}} [\tan^{-1}\sqrt{3} - \tan^{-1} 0]$

$= \dfrac{\pi^2}{6\sqrt{3}}\ units^3$

End of Question 13.

Question 14 : Use new writing book for this question **15 Marks**

a) i. Show $\sin(x-y)\sin(x+y) = \sin^2 x - \sin^2 y$. **2**

Sample answer:

$$\begin{aligned}\sin(x-y)\sin(x+y) &= \frac{1}{2}\left[\cos\left[(x-y)-(x+y)\right]-\cos\left[(x-y)+(x+y)\right]\right]\\ &= \frac{1}{2}\left[\cos(-2y)-\cos(2x)\right]\\ &= \frac{1}{2}\left[\cos(2y)-\cos(2x)\right]\\ &= \frac{1}{2}\left[\left(1-2\sin^2 y\right)-\left(1-2\sin^2 x\right)\right]\\ &= \sin^2 x-\sin^2 y\end{aligned}$$

ii. Solve $\sin^2 3x - \sin^2 x = \sin 4x$ for $0 \leq x \leq \pi$ **2**

Sample answer:

From (i) $\sin^2 3x-\sin^2 x=\sin(3x-x)\sin(3x+x)=\sin 2x\sin 4x$

$\sin 2x\sin 4x=\sin 4x$ *for* $0\le x\le \pi$

$\sin 2x\sin 4x-\sin 4x=0$

$\sin 4x(\sin 2x-1)=0$

$\sin 4x=0 \Rightarrow 4x=0,\pi,2\pi,3\pi,4\pi$

$x=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi$

$\sin 2x=1 \Rightarrow 2x=\frac{\pi}{2} \Rightarrow x=\frac{\pi}{4}$

$\therefore x=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi$

b) The population $P(t)$ of bacteria in a petri dish is modelled by the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{6}\left(1 - \frac{P}{8000}\right)$$ where $P(0)=P_0$ and t is the time in hours.

i. If the initial population P_0 is 1000 bacteria, show that $P(t) = \dfrac{8000}{1 + 7e^{-\frac{t}{6}}}$ **3**

$$\frac{dP}{dt} = \frac{P}{6}\left(1 - \frac{P}{8000}\right)$$
$$\frac{dt}{dP} = \frac{48000}{P(8000 - P)}$$
$$\int dt = 6\int \frac{8000}{P(8000 - P)}dP$$
$$\frac{1}{6}\int dt = \int\left(\frac{1}{P} + \frac{1}{8000 - P}\right)dP \text{ using } \frac{Q}{P(Q - P)} = \frac{1}{P} + \frac{1}{Q - P}$$
$$\frac{1}{6}t + c = ln|P| - ln|8000 - P|$$
$$\frac{t}{6} + c = ln\left|\frac{P}{8000 - P}\right|$$
$$\left|\frac{P}{8000 - P}\right| = e^c e^{\frac{t}{6}}$$
$$\frac{P}{8000 - P} = \pm e^c e^{\frac{t}{6}}$$
$$\frac{P}{8000 - P} = Ae^{\frac{t}{6}}\text{, where } A = \pm e^c$$
When $t = 0,\ P = 1000$:
$$\frac{1000}{8000 - 1000} = Ae^0$$
$$\therefore A = \frac{1}{7}$$
$$\therefore \frac{P}{8000 - P} = \frac{1}{7}e^{\frac{t}{6}}$$
$$\therefore P = \frac{8000}{1 + 7e^{-\frac{t}{6}}}$$

(You may use the fact that $\dfrac{Q}{P(Q - P)} = \dfrac{1}{P} + \dfrac{1}{Q - P}$)

ii. If instead the initial population P_0 were 12000 bacteria, describe what would have happened to the population as t increases. **1**

Sample answer:

The carrying capacity for the population is 8000. If $P(0) > 8000$, such as 12000, the population will decay and approach the asymptotic population size of 8000 as time increases.

Alternatively: The equation for $\frac{dP}{dt}$ shows that when $P = 12000$, $\frac{dP}{dt} < 0$ (population will decrease). As *P* decreases and approaches 8000, $\frac{dP}{dt}$ approaches zero. Thus, the population will decrease and will approach 8000.

c) The vectors $\underset{\sim}{\mathbf{p}} = \begin{pmatrix} a \\ 3 \end{pmatrix}$ and $\underset{\sim}{\mathbf{q}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are parallel.

i. Find a. **1**

$$Parallel \Rightarrow \lambda \begin{pmatrix} a \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$3\lambda = 4 \Rightarrow \lambda = \frac{4}{3}$$

$$\frac{4}{3}a = 2$$

$$a = \frac{3}{2}$$

ii. The vector $\underset{\sim}{\mathbf{r}} = \begin{pmatrix} x \\ y \end{pmatrix}$ is perpendicular to vector $\underset{\sim}{\mathbf{q}}$ and $\left|\underset{\sim}{\mathbf{q}} - \underset{\sim}{\mathbf{r}}\right| = \sqrt{65}$.

Find the possible values of x and y. **3**

$$\text{Perpendicular} \Rightarrow \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$2x + 4y = 0$$

$$x = -2y$$

$$\overset{r}{q} - \overset{r}{r} = \begin{pmatrix} 2-x \\ 4-y \end{pmatrix} = \begin{pmatrix} 2+2y \\ 4-y \end{pmatrix}$$

$$\left|\overset{r}{q} - \overset{r}{r}\right| = \sqrt{65} \Rightarrow \sqrt{(2+2y)^2 + (4-y)^2} = \sqrt{65}$$

squaring

$$4 + 8y + 4y^2 + 16 - 8y + y^2 = 65$$

$$5y^2 + 20 = 65$$

$$y^2 = 9$$

$$y = \pm 3$$

$$\therefore x = 6, y = -3 \text{ or } x = -6, y = 3$$

d) Using the expansion of $(1+x)^{2n}$ and given that **3**

$${}^{2n}\mathbf{C}_1 + 2({}^{2n}\mathbf{C}_2) + 3({}^{2n}\mathbf{C}_3) + \ldots + 2n({}^{2n}\mathbf{C}_{2n}) = n \times 2^{2n}$$

Show that

$$2({}^{2n}\mathbf{C}_1) + 3({}^{2n}\mathbf{C}_2) + 4({}^{2n}\mathbf{C}_3) + \ldots + (2n+1)({}^{2n}\mathbf{C}_{2n}) = (n+1) \times 2^{2n} - 1$$

Sample answer:

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots\ldots + + \binom{2n}{2n}x^n$$

substitute $x = 1$

$$\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \ldots\ldots + + \binom{2n}{2n} = 2^{2n}$$

$$\binom{2n}{1} + \binom{2n}{2} + \ldots\ldots + \binom{2n}{2n} = 2^{2n} - \binom{n}{0}$$

$$\binom{2n}{1} + \binom{2n}{2} + \ldots\ldots + + \binom{2n}{2n} = 2^{2n} - 1$$

$$2\binom{2n}{1} + 3\binom{2n}{2} + \ldots\ldots + (2n+1)\binom{2n}{2n} = \left[\binom{2n}{1} + 2\binom{2n}{2} + \ldots\ldots + 2n\binom{2n}{2n}\right] + \left[\binom{2n}{1} + \binom{2n}{2} + \ldots\ldots + + \binom{2n}{2n}\right]$$

$$= n \times 2^{2n} + 2^{2n} - 1$$

$$= (n+1) \times 2^{2n} - 1$$

End of Examination

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